

Hölder continuity of harmonic functions for Hunt processes with Green function

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Abstract

Let (X, \mathcal{W}) be a balayage space, $1 \in \mathcal{W}$, or – equivalently – let \mathcal{W} be the set of excessive functions of a Hunt process on a locally compact space X with countable base such that \mathcal{W} separates points, every function in \mathcal{W} is the supremum of its continuous minorants and there exist strictly positive continuous $u, v \in \mathcal{W}$ such that $u/v \rightarrow 0$ at infinity. We suppose that there is a Green function $G > 0$ for X , a metric ρ on X and a decreasing function $g: [0, \infty) \rightarrow (0, \infty]$ having the doubling property and a mild upper decay such that $G \approx g \circ \rho$ and the capacity of balls of radius r is approximately $1/g(r)$.

It is shown that bounded harmonic functions are Hölder continuous, if the constant function 1 is harmonic and jumps out of balls admit a polynomial estimate. The latter is proven if scaling invariant Harnack inequalities hold.

Keywords: Hunt process; balayage space; Lévy process; Green function; harmonic function, Hölder continuity

MSC: 31D05, 60J25, 60J45, 60J65, 60J75.

1 Setting

Our basic setting will be as in [5] to which we refer for further details and citations: Let X be a locally compact space with countable base. Let $\mathcal{C}(X)$ denote the set of all continuous real functions on X and let $\mathcal{B}(X)$ be the set of all Borel measurable numerical functions on X . The set of all (positive) Radon measures on X will be denoted by $\mathcal{M}(X)$.

Moreover, let \mathcal{W} be a convex cone of positive lower semicontinuous numerical functions on X such that $1 \in \mathcal{W}$ and (X, \mathcal{W}) is a balayage space. In particular, the following holds:

(C) \mathcal{W} separates the points of X ,

$$w = \sup\{v \in \mathcal{W} \cap \mathcal{C}(X) : v \leq w\} \quad \text{for every } w \in \mathcal{W},$$

and there are strictly positive $u, v \in \mathcal{W} \cap \mathcal{C}(X)$ such that $u/v \rightarrow 0$ at infinity.

There exists a Hunt process \mathfrak{X} on X such that \mathcal{W} is the set $E_{\mathbb{P}}$ of excessive functions for the transition semigroup $\mathbb{P} = (P_t)_{t \geq 0}$ of \mathfrak{X} , that is,

$$\mathcal{W} = \{v \in \mathcal{B}^+(X) : \sup_{t \geq 0} P_t v = v\}.$$

(Conversely, given any sub-Markov right-continuous semigroup $\mathbb{P} = (P_t)_{t \geq 0}$ on X such that (C) is satisfied by its convex cone $E_{\mathbb{P}}$ of excessive functions, $(X, E_{\mathbb{P}})$ is a balayage space, and \mathbb{P} is the transition semigroup of a Hunt process.)

For every subset A of X , we have reduced functions R_u^A , $u \in \mathcal{W}$, and reduced measures ε_x^A , $x \in X$, defined by

$$R_u^A := \inf\{v \in \mathcal{W} : v \geq u \text{ on } A\} \quad \text{and} \quad \int u d\varepsilon_x^A = R_u^A(x).$$

If A is a Borel set, then

$$(1.1) \quad R_1^A(x) = P^x[T_A < \infty], \quad x \in X,$$

where $T_A(\omega) := \inf\{t \geq 0 : X_t(\omega) \in A\}$ and, for every Borel set B in X ,

$$\varepsilon_x^A(B) = P^x[X_{T_A} \in B; T_A < \infty].$$

For every open set U in X , let $\mathcal{H}^+(U)$ denote the set of all functions $h \in \mathcal{B}^+(X)$ which are *harmonic on U* (in the sense of [1]), that is, such that $h|_U \in \mathcal{C}(U)$ and

$$(1.2) \quad \varepsilon_x^{V^c}(h) := \int h d\varepsilon_x^{V^c} = h(x) \quad \text{if } V \text{ is open and } x \in V \subset \subset U.$$

Analogously, we define the set $\mathcal{H}_b(U)$ of all bounded functions which are harmonic on U and note that, given $h \in \mathcal{B}_b(X)$, already (1.2) implies that $h|_U \in \mathcal{C}(U)$.

We have the following sheaf property: If U_i , $i \in I$, are open sets in X , then

$$\bigcap_{i \in I} \mathcal{H}^+(U_i) = \mathcal{H}^+\left(\bigcup_{i \in I} U_i\right).$$

In fact, given an open set U in X , a function $h \in \mathcal{B}^+(X)$ which is continuous on U is already contained in $\mathcal{H}^+(U)$, if, for every $x \in U$, there exists a fundamental system of relatively compact open neighborhoods V of x in U such that $\varepsilon_x^{V^c}(h) = h(x)$.

Let us fix once and for all a point $x_0 \in X$. In order to discuss Hölder continuity of bounded harmonic functions at x_0 we suppose the following (for an additional, later assumption see (3.1)).

ASSUMPTION 1.1. *We have a Borel measurable function $G: X \times X \rightarrow (0, \infty]$ with $G = \infty$ on the diagonal such that the following hold.*

- (i) *For every $y \in X$, $G(\cdot, y)$ is a potential which is harmonic on $X \setminus \{y\}$.*
- (ii) *For every potential p on X , there exists $\mu \in \mathcal{M}(X)$ such that*

$$p = G\mu := \int G(\cdot, y) d\mu(y).$$

- (iii) *There are constants $a_0 \geq 0$ and $c_1 \geq 1$ such that, for all $a > a_0$,*

$$R_1^{\{G(\cdot, x_0) > a\}} \geq c_1^{-1} \frac{G(\cdot, x_0)}{a} \quad \text{on } \{G(\cdot, x_0) \leq a\}.$$

(iv) *There is a metric ρ for X , a decreasing function $g: [0, \infty) \rightarrow (0, \infty]$ and constants $c, c_D \in [1, \infty)$, $\alpha_0, \eta_0 \in (0, 1)$ such that, for every $r > 0$,*

$$g(r/2) \leq c_D g(r), \quad g(r) \leq \eta_0 g(\alpha_0 r) \quad \text{and} \quad c^{-1} g \circ \rho \leq G \leq c g \circ \rho.$$

REMARKS 1.2. 1. Property (iv) implies that $g(0) = \lim_{r \rightarrow 0} g(r) = \infty$ and that, for *any* $\eta > 0$, there exists $\alpha \in (0, 1)$ such that $g(r) \leq \eta g(\alpha r)$ for every $r > 0$ (choose $k \in \mathbb{N}$ with $\eta_0^k \leq \eta$ and take $\alpha := \alpha_0^k$).

2. For applications leading to intrinsic Hölder continuity of bounded harmonic functions we recall the following (see [4, Appendix]). Suppose that we have a function $G: X \times X \rightarrow (0, \infty]$ with $G = \infty$ on the diagonal such that (i) holds, each potential $G(\cdot, y)$, $y \in X$, is bounded at infinity, and G has the triangle property, that is, there exists $C > 0$ such that

$$G(x, z) \wedge G(y, z) \leq C G(x, y), \quad x, y, z \in X.$$

Then there exist a metric d for X and $\gamma > 0$ such that

$$G \approx d^{-\gamma},$$

which clearly implies (iv) with $g(r) := r^{-\gamma}$ and d in place of ρ (conversely, (iv) implies that G has the triangle property).

Inner capacities for open sets U in X are defined by

$$(1.3) \quad \text{cap}_* U := \sup \{ \|\mu\| : \mu \in \mathcal{M}(X), \mu(X \setminus U) = 0, G\mu \leq 1 \}$$

and outer capacities for arbitrary sets A in X by

$$(1.4) \quad \text{cap}^* A := \inf \{ \text{cap}_* U : U \text{ open neighborhood of } A \}.$$

Obviously, $\text{cap}^* A = \text{cap}_* A$, if A is open. If $\text{cap}_* A = \text{cap}^* A$, we may simply write $\text{cap} A$ and speak of the capacity of A .

For $r > 0$, let

$$B(r) := \{x \in X : \rho(x, x_0) < r\},$$

and let R_0 denote the supremum of all $r > 0$ such that $B(r)$ is relatively compact and $c g(r) > a_0$. Then $0 < R_0 \leq \infty$. Let

$$c_0 := c^3 c_D c_1.$$

LEMMA 1.3. *For all $0 < r < R_0$,*

$$c_0^{-1} g(r)^{-1} \leq \text{cap} B(r) \leq c g(r)^{-1}.$$

Proof. The second inequality is part of [4, Proposition 1.7] (and holds for all $r > 0$).

To prove the first inequality we fix $0 < r < R_0$ and note first that, by (iv), $G(x, x_0) > c g(r)$ implies that $\rho(x, x_0) < r$, and hence, by (iii),

$$R_1^{B(r)} \geq R_1^{\{G(\cdot, x_0) > c g(r)\}} \geq c_1^{-1} \frac{G(\cdot, x_0)}{c g(r)} \quad \text{on } \{G(\cdot, x_0) \leq c g(r)\} \subset B(r)^c.$$

By [4, Proposition 1.10, (b)], this implies that $\text{cap} B(r) \geq c_0^{-1} g(r)^{-1}$. \square

Let us note that conversely, by [4, Proposition 1.10,(a)], any estimate $\text{cap } B(r) \geq C^{-1}g(r)^{-1}$ implies that $R_1^{B(r)} \geq (c^2 c_D C)^{-1} G(\cdot, x_0)/g(r)$ on $X \setminus B(r)$ (which in turn implies (iii)).

LEMMA 1.4. *Let $\beta \in (0, 1)$ such that $g(r) \leq (2cc_0)^{-1}g(\beta r)$ for all $r > 0$. Then, for every $0 < r < R_0$, the shell $S(r) := B(r) \setminus B(\beta r)$ satisfies $\text{cap}^* S(r) \geq (2c_0)^{-1}g(r)^{-1}$.*

Proof. By Lemma 1.3 and the subadditivity of cap^* ,

$$c_0^{-1}g(r)^{-1} \leq \text{cap } B(r) \leq \text{cap}^* S(r) + \text{cap } B(\beta r),$$

where $\text{cap } B(\beta r) \leq cg(\beta r)^{-1} \leq (2c_0)^{-1}g(r)^{-1}$, by Lemma 1.3. \square

2 Control of jumps having Harnack inequalities

Let us observe first that the probabilistic statements and proofs in this section can be replaced by analytic ones using that, for all Borel sets A, B in an open set U (where, as usual $\tau_U := T_{U^c}$),

$$P^x[X_{T_A} \in B; T_A < \tau_U] = \varepsilon_x^{A \cup U^c}(B)$$

(see [1, VI.2.9]) and, for all Borel sets A, B in X with $B \subset A$,

$$(2.1) \quad \varepsilon_x^B = \varepsilon_x^A|_B + (\varepsilon_x^A|_{B^c})^B.$$

(If $x \in B$, then (2.1) holds trivially. If $x \notin B$ and $u \in \mathcal{W}$, then, by [1, VI.9.1],

$$R_u^B(x) = R_u^B(x) = \int R_u^B d\varepsilon_x^A = \int_B u d\varepsilon_x^A + \int_{B^c} \hat{R}_u^B d\varepsilon_x^A.)$$

Next we establish a useful estimate for the hitting of a union of two sets.

LEMMA 2.1. *Let U, V be open sets in X such that the exit time τ_V is finite almost surely. Let A be a Borel set in U , B a Borel set in $V \setminus U$ and let $x \in U$ and $\kappa \geq 0$ such that $P^y[T_B \geq \tau_V] \geq \kappa P^x[T_B \geq \tau_V]$ for every $y \in U \cap \bar{A}$. Then*

$$P^x[T_{A \cup B} \geq \tau_V] \leq (1 - \kappa P^x[T_A < \tau_U]) P^x[T_B \geq \tau_V].$$

Proof. Defining $E := [T_A \geq \tau_U]$ and $F := [T_B \geq \tau_V]$ we have

$$P^x[T_{A \cup B} \geq \tau_V] = P^x[T_A \geq \tau_V, T_B \geq \tau_V] \leq P^x(E \cap F) = P^x(F) - P^x(F \setminus E).$$

Clearly, $F \cap [\tau_V < \infty] = [X_{T_{B \cup V^c}} \in V^c]$ and $X_{T_{B \cup V^c}} = X_{T_{B \cup V^c}} \circ \theta_{T_A}$ on $[T_A < \tau_U]$. Since $X_{T_A} \in U \cap \bar{A}$ on $[T_A < \tau_U]$, the strong Markov property hence yields that

$$P^x(F \setminus E) = \int_{[T_A < \tau_U]} P^{X_{T_A}}(F) dP^x \geq \inf_{y \in \bar{A} \cap U} P^y(F) \cdot P^x[T_A < \tau_U].$$

By our assumption, the proof is completed combining the two estimates. \square

As in [5] let us define

$$\eta := (2c^3 c_D^2)^{-1}.$$

By Remark 1.2, there exists $0 < \alpha < 1/2$ such that $g(r) \leq \eta g(\alpha r)$ for every $r > 0$. Taking $\alpha_1 := \alpha/2$ we have

$$(2.2) \quad g((1 - 2\alpha_1)r) \leq \eta g(\alpha_1 r) \quad \text{for every } r > 0.$$

Of course, (2.2) still holds if we replace α_1 by any $\alpha \in (0, \alpha_1)$.

Let us recall the following estimate for the probability of hitting a set before leaving a large ball (see [5, Proposition 3.2]).

PROPOSITION 2.2. *For all $r > 0$, $0 < \alpha \leq \alpha_1$, $x \in B(2\alpha r)$ and Borel sets A in $B(2\alpha r)$,*

$$(2.3) \quad P^x[T_A < \tau_{B(r)}] \geq \eta g(\alpha r) \text{cap}^*(A).$$

After these preparations we arrive at the main result of this section.

PROPOSITION 2.3. *Suppose that the constant function 1 is harmonic and there exist $0 < \alpha \leq \alpha_1$ and $K > 0$ such that, for $0 < r < R_0$,*

$$(2.4) \quad \sup h(B(\alpha r)) \leq K \inf h(B(\alpha r)) \quad \text{for every } h \in \mathcal{H}_b^+(B(r)).$$

Let $a := 1 - (2c_0 K)^{-1} \eta$. Then, for all $0 < r < R_0$ and $m = 0, 1, 2, \dots$,

$$(2.5) \quad \varepsilon_x^{B(\alpha^{2m}r)^c}(B(r)^c) \leq a^m \quad \text{for every } x \in B(\alpha^{2m}r).$$

Proof. Of course, (2.5) holds trivially if $m = 0$. Let us fix $0 < r < R_0$. For $m = 0, 1, 2, \dots$, we define

$$B_m := B(\alpha^m r) \quad \text{and} \quad S_m := B_0 \setminus B_m.$$

Then, for every $x \in B_m$,

$$(2.6) \quad \varepsilon_x^{B_m^c}(B_0^c) = P^x[T_{S_{2m}} \geq \tau_{B_0}].$$

For the moment, let us fix $m \in \mathbb{N}$ and define

$$A := B_{2m-1} \setminus B_{2m}, \quad U := B_{2m-2}, \quad V := B_0, \quad B := V \setminus U = S_{2m-2}.$$

By Proposition 2.2 and Lemma 1.4, for all $x \in B_{2m-1}$,

$$P^x[T_A < \tau_U] \geq \eta g(\alpha^{2m-1}r) \text{cap}^*(A) \geq (2c_0)^{-1} \eta.$$

The function $y \mapsto P^y[T_B \geq \tau_V] = \varepsilon_y^{B \cup V^c}(V^c)$ is harmonic on U , by [1, VI.2.10], and hence, by (2.4),

$$P^y[T_B \geq \tau_V] \leq K P^x[T_B \geq \tau_V] \quad \text{for all } y \in \overline{A}.$$

So, by Lemma 2.1, for every $x \in B_{2m}$,

$$P^x[T_{S_{2m}} \geq \tau_V] \leq P^x[T_{A \cup B} \geq \tau_V] \leq a P^x[T_B \geq \tau_V] = a P^x[T_{S_{2m-2}} \geq \tau_V].$$

In view of (2.6), the proof of (2.5) is completed by induction. \square

3 Hölder continuity

In addition to Assumption 1.1 and harmonicity of the constant function 1, let us suppose that there exist $a_0, \gamma \in (0, 1)$ and $C_0 \geq 1$ such that, for all $0 < r < R_0$, $m \in \mathbb{N}$ and $x \in B(\gamma^m r)$,

$$(3.1) \quad \varepsilon_x^{B(\gamma^m r)^c}(B(r)^c) \leq C_0 a_0^m.$$

REMARKS 3.1. 1. By Proposition 2.5, (3.1) holds if we have the Harnack inequalities (2.4). In [6, Theorem 4.1], Hölder continuity is obtained assuming (more strongly) a version of Harnack inequalities for bounded functions which are harmonic and positive on $B(r)$, but may be negative on the complement.

2. If \mathfrak{X} is a diffusion, that is, if the reduced measures $\varepsilon_x^{U^c}$ for open sets U and $x \in U$ are supported by the boundary of U , then (3.1) holds trivially.

3. It is known that (3.1) holds for many Lévy processes (see [2, Corollary 2]).

To obtain a suitable Hölder exponent β we first define $\delta := (12c_0)^{-1}\eta$ and fix $b \in (1, \sqrt{3/2})$, $a \in (0, 1/3)$ such that

$$(3.2) \quad b^2(1 - 3\delta) < 1 - 2\delta \quad \text{and} \quad ab^3(1 - ab)^{-1} < \delta.$$

Then we choose $k \in \mathbb{N}$ such that $a_0^k < C_0^{-1}a$, and define

$$(3.3) \quad \alpha := \alpha_1 \wedge \gamma^k, \quad \beta := (\ln b) \cdot \left(\ln \frac{1}{\alpha}\right)^{-1}.$$

By (3.1) and our choice of α , for all $0 < r < R_0$, $m = 0, 1, 2, \dots$ and $x \in B(\alpha^m r)$,

$$(3.4) \quad \varepsilon_x^{B(\alpha^m r)^c}(B(r)^c) \leq a^m.$$

THEOREM 3.2. *For all $0 < r < R_0$, $h \in \mathcal{H}_b(B(r))$ and $x \in B(r)$,*

$$(3.5) \quad |h(x) - h(x_0)| \leq 3\|h\|_\infty \left(\frac{\rho(x, x_0)}{\alpha r} \right)^\beta.$$

Proof. Except for using capacity instead of a certain measure, we may follow rather closely the proof of [7, Theorem 1.4].

Let $0 < r < R_0$ and $h \in \mathcal{H}_b(B)$, $\|h\|_\infty = 1$. For $n = 0, 1, 2, \dots$ let

$$B_n := B(\alpha^n r), \quad m_n := \inf h(\overline{B_n}), \quad M_n := \sup h(\overline{B_n}).$$

We claim that

$$(3.6) \quad M_n - m_n \leq s_n := 3b^{-n}.$$

Clearly, (3.6) holds trivially for $n = 0, 1, 2$, since $M_n - m_n \leq 2$ and $b^2 < 3/2$. Suppose that (3.6) holds for some $n \in \mathbb{N}$, $n \geq 2$. Given $\varepsilon > 0$, we may choose points $x, y \in B_{n+1}$ such that

$$h(x) > M_{n+1} - \varepsilon \quad \text{and} \quad h(y) < m_{n+1} + \varepsilon.$$

We intend to prove that

$$h(x) - h(y) \leq s_{n+1}.$$

To that end we may assume without loss of generality that the closed set

$$A := \left\{ z \in \overline{B}_{n+1} : h(z) \leq \frac{m_n + M_n}{2} \right\}$$

satisfies $\text{cap}^* A \geq (1/2) \text{cap} \overline{B}_{n+1}$. Indeed, otherwise we replace h by $-h$ and exchange the roles of x and y . Let

$$\mu := \varepsilon_x^{A \cup B_n^c}.$$

Since $1, h \in \mathcal{H}_b(B_0)$, we know that μ is a probability measure and

$$(3.7) \quad h(x) - h(y) = \int (h - h(y)) d\mu.$$

The measure μ is supported by $A \cup B_n^c$. Clearly,

$$\int_A (h - h(y)) d\mu \leq \left(\frac{m_n + M_n}{2} - m_n \right) \mu(A) \leq \frac{1}{2} s_n \mu(A) \leq \frac{1}{2} s_{n-1} \mu(A),$$

where, by Proposition 2.2 and Lemma 1.3,

$$\mu(A) \geq \eta g(\alpha^{n+1} r) \text{cap}^* A \geq \eta g(\alpha^{n+1} r) \cdot (2c_0)^{-1} g(\alpha^{n+1} r)^{-1} = 6\delta.$$

Since $\mu(B_n^c) = 1 - \mu(A)$, we have

$$\int_{B_{n-1} \setminus B_n} (h - h(y)) d\mu \leq s_{n-1} (1 - \mu(A)).$$

Combining the three estimates we obtain that

$$(3.8) \quad \int_{A \cup (B_{n-1} \setminus B_n)} (h - h(y)) d\mu \leq s_{n-1} \left(1 - \frac{1}{2} \mu(A) \right) \leq s_{n+1} b^2 (1 - 3\delta).$$

Finally,

$$(3.9) \quad \int_{B_{n-1}^c} (h - h(y)) d\mu \leq 2\mu(B_0^c) + \sum_{j=0}^{n-2} s_j \mu(B_j \setminus B_{j+1}).$$

By (2.1) and (3.4), for every $1 \leq m \leq n$,

$$\mu(B_m^c) \leq \varepsilon_x^{B_n^c}(B_m^c) \leq a^{n-m}.$$

So, by (3.2), $2\mu(B_0^c) \leq 2a^n \leq s_{n+1}\delta$ (note that $ab < \delta$) and

$$\sum_{j=0}^{n-2} s_j \mu(B_j \setminus B_{j+1}) \leq \sum_{j=0}^{n-2} s_j \mu(B_{j+1}^c) \leq 3 \sum_{j=0}^{n-2} b^{-j} a^{n-(j+1)} = s_{n+1} s,$$

where,

$$s = b^{n+1} \sum_{j=0}^{n-2} b^{-j} a^{n-(j+1)} = b^2 \sum_{j=0}^{n-2} (ab)^{n-(j+1)} \leq \frac{ab^3}{1-ab} \leq \delta.$$

Having (3.7), the estimates (3.8) and (3.9) hence yield that $h(x) - h(y) \leq s_{n+1}$. Thus $M_{n+1} - m_{n+1} \leq s_{n+1}$, since $\varepsilon > 0$ was arbitrary, and the inductive step for (3.6) is finished.

Given $x \in B_0 \setminus \{x_0\}$, there exists $n \geq 0$ such that $x \in B_n \setminus B_{n+1}$, and therefore, by (3.3),

$$|h(x) - h(x_0)| \leq 3b^{-n} = 3\alpha^{n\beta} \leq 3\left(\frac{\rho(x, x_0)}{\alpha r}\right)^\beta$$

completing the proof. \square

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